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A METRIC MINIMAL FLOW WHOSE ENVELOPING SEMIGROUP CONTAINS FINITELY MANY MINIMAL IDEALS IS PI

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ABSTRACT

We show that a minimal metric flow X whose enveloping semigroup $E(X)$, contains finitely many minimal ideals, is a PI-flow; i.e. X has a proximal extension X' which can be built by iterating proximal and isometric extensions (starting with the trivial one point flow). An example is given which shows that the converse theorem does not hold. Finally, we show that if X is a minimal, non-trivial, metric, weakly mixing flow and the group action is nilpotent then $E(X)$ contains infinitely many minimal ideals.

In this note we generalize a result which appears in [1]. This result ([1, Prop. 8.8]) states that if X is a metric minimal flow whose proximal relation is an equivalence relation then X is a PI-flow. Now it is easy to see that for a minimal flow X, the condition "Proximal is an equivalence relation" is equivalent to the condition "$E(X)$, the enveloping semigroup of X, contains a unique minimal ideal". Our result is that a metric minimal flow X is PI whenever $E(X)$ contains only finitely many minimal ideals.

We shall use the machinery and technique developed in [2] and [1] to prove this result (Theorem 1 below). Also we shall use freely the notations and theorems introduced in [1]. However, in order to make at least the statements of Theorem 1 and 2 independent of other sources, we give the following definitions (see [3] for a general account).

A *flow* is a triple which consists of a compact Hausdorff space X, a locally compact topological group T and a jointly continuous function (written $(x,t)\rightarrow xt$) of $X \times T$ onto X which satisfies the conditions $(xt)s = x(ts)$ and $xe = x$ for all $x \in X$ and $s, t \in T$, where e is the identity element of T. We denote a flow by (X, T) or sometimes just X.

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A flow is *minimal* if it contains no proper closed invariant non-empty subsets. A *homomorphism* (or *extension*) $X \xrightarrow{\phi} Y$ of a minimal flow X onto a minimal flow Y is a continuous map which satisfies $\phi(xt) = \phi(x)t$ for all $x \in X$ and $t \in T$.

A pair of points x_1, x_2 in a flow X is called *proximal* if there exists a net $\{t_i\}$ in T such that $\lim t_i x_1 = \lim t_i x_2$. The subset of $X \times X$ defined by

$$P = P(X) = \{(x_1, x_2): x_1 \text{ and } x_2 \text{ are proximal}\}$$

is called the *proximal relation* on X. A homomorphism $X \xrightarrow{\phi} Y$ is called *proximal* if all pairs of points x_1, x_2 in X such that x_1, $x_2 \in \phi^{-1}(y)$, for some $y \in Y$, are proximal. The homomorphism $X \xrightarrow{\phi} Y$ is called *distal* if whenever x_1, $x_2 \in \phi^{-1}(y)$ for some $y \in Y$ and $x_1 \neq x_2$ then $(x_1, x_2) \notin P$. $X \xrightarrow{\phi} Y$ is an *isometric extension* ([4]) if there exists a real valued function $\rho(x_1, x_2)$ defined for all pairs in the subset $\{\phi^{-1}(y) \times \phi^{-1}(y): y \in Y\}$ of $X \times X$ such that

(a) $\rho(x_1, x_2)$ is continuous on its domain,

(b) for each $y \in Y$, ρ restricted to $\phi^{-1}(y) \times \phi^{-1}(y)$ is a metric on $\phi^{-1}(y)$, such that all the metric spaces $\phi^{-1}(y)$ $(y \in Y)$ are isometric,

(c) $\rho(x_1 t, x_2 t) = \rho(x_1, x_2)$ for all $t \in T$ and (x_1, x_2) in the domain of ρ.

Clearly an isometric extension is distal.

We say that a minimal flow is *strictly* PI if there exists an ordinal η and for each ordinal $\alpha \leqq \eta$ a flow X_α such that

(i) X_0 is the trivial one point flow and X_η is X,

(ii) for each $\alpha \leqq \eta$ there is a homomorphism $X_{\alpha+1} \xrightarrow{\phi_{\alpha+1}} X_\alpha$ which is either proximal or isometric,

(iii) if $\alpha \leqq \eta$ is a limit ordinal then X_α is isomorphic to a minimal subset of the projective limit $\Delta \subseteq \Pi\{X_\beta : \beta < \alpha\}$ which is defined by

$$\Delta = \{(x_0, x_1, \cdots, x_\beta, \cdots): x_i = \phi_{i+1}(x_{i+1})\}.$$

X is a PI-*flow* if there exists a proximal extension X' of X which is strictly PI.

Given a flow (X, T) we can consider T, whose action we assume to be effective, as a subset of the space X^X of all mappings of X into itself. The closure of this subset in X^X is a semigroup of transformations of X into itself (usually far from being continuous or even measurable) which is called the *enveloping semigroup* of X and is denoted by $E(X)$. The semigroup multiplication by elements of T on the right makes $E(X)$ a flow of the group T (as a discrete group). It is easy to see that the minimal right ideals of the semigroup $E(X)$ coincide with the minimal subsets of the flow $(E(X), T)$. All these minimal sets are isomorphic. Since only minimal right ideals are of interest we refer to them as *minimal ideals.*

A flow (X, T) is *weakly mixing* if whenever A, B, C and D are non-empty open subsets of X then there exists $t \in T$ such that in $X \times X$, $(A \times B)t \cap (C \times D) \neq \emptyset$.

We can state now our first theorem.

THEOREM 1. *Let X be a minimal flow where X is a metric space. Suppose $E(X)$ contains finitely many minimal ideals; then X is a* PI*-flow.*

PROOF. We proceed by steps.

(a) We let βT be the Stone-Čech compactification of the discrete group T, M a fixed minimal ideal in βT, J the set of idempotents in M, u a fixed element of J and $G = Mu$. We choose $x_0 \in X$ such that $x_0 u = x_0$ and we let $\mathscr{A}$ be the subalgebra of $\mathfrak{A}(u)$ which corresponds to the pointed flow (X, x_0) (i.e.

$$\mathscr{A} = \{f(t) = F(x_0 t): F \text{ is a real valued continuous function on } X\}).$$

We put

$$A = \mathfrak{G}(X, x_0) = \mathfrak{G}(\mathscr{A}) = \{\alpha \in G : x_0\alpha = x_0\}$$

$$= \{\alpha \in G : \alpha \mid_{\mathscr{A}} = u \mid_{\mathscr{A}}\}.$$

(b) By [1, 7.5 and 7.7.2] there exists a metric minimal strictly PI-flow Y_∞ with a base point y_∞ such that

(1) the orbit closure of $x_\infty = (x_0, y_\infty) \in X \times Y_\infty$ is a minimal flow, X_∞,

(2) the projection map $X_\infty \xrightarrow{\psi} X$ is a proximal homomorphism,

(3) $AH(F, \tau) = F$ where

$$F = \mathfrak{G}(Y_\infty, y_\infty) = \{\alpha \in G : y_\infty \alpha = y_\infty\},$$

(4) the projection $X_\infty \xrightarrow{\phi} Y_\infty$ is a RIC-extension (see [1, Sec. 5]).

Notice that since ψ is proximal $A = \mathfrak{G}(X_\infty, x_\infty) = \mathfrak{G}(X, x_0)$. Also since X_∞ is an extension of Y_∞, $A \subseteq F$. Now since ϕ is RIC, for each $y \in Y_\infty$

$$\phi^{-1}(y) = x_\infty F \circ p \text{ where } p \in M \text{ is such that } y_\infty p = y.$$

Thus if we show that $F = A$ then

$$\phi^{-1}(y) = x_\infty A \circ p = \{x_\infty\} \circ p = x_\infty p$$

and ϕ is one-to-one; i.e. X_∞ is isomorphic to the strictly PI-flow Y_∞ and it will follow then that X is PI. Thus all we have to show is that $A = F$.

(c) The group F acts on M (on the left) and so induces a flow (F, M). F is a subset of M and if $\bar{F}$ denotes the closure of F in M then $\bar{F}$ is an F-invariant closed subset of M. It is easy to see that there exists an idempotent $w \in \bar{F}$ such that w is an F-almost-periodic point (i.e. $\overline{Fw}$ is an F-minimal set). We fix such a w and we let

$$K = \{p \in \overline{Fw}: x_\infty p \text{ and } x_\infty \text{ are proximal}\}.$$

By [1, Lemma 8.5] K is a residual subset of $\overline{Fw}$. (This is the deepest part of the proof and is already proved in [2].)

(d) Let $I_1, \cdots, I_n$ be the finite set of minimal ideals in $E(X)$. Let $M_1, \cdots, M_n$ be n minimal ideals in βT which are mapped onto $I_1, \cdots, I_n$ respectively. Let J_i be the set of idempotents in M_i; then it is well known that

$$\begin{aligned} P(x_0) &= \{x \in X: x \text{ is proximal to } x_0\} \\ &= \bigcup_{i=1}^{n} x_0 J_i . \end{aligned}$$

Define

$$K_i = \{p \in \overline{Fw}: x_0 p \in x_0 J_i\} \qquad i = 1, 2, \cdots, n.$$

We claim that $K = \bigcup_{i=1}^{n} K_i$. Indeed if $p \in \overline{Fw}$ then since $\overline{Fw} \subseteq \bar{F}$, $y_\infty p = y_\infty$ so that $x_0 p$ is proximal to x_0 implies that $x_\infty p = (x_0, y_\infty)p = (x_0 p, y_\infty)$ is proximal to $x_\infty = (x_0, y_\infty)$. This shows that $K_i \subseteq K$. Conversely if $p \in K$ then $x_\infty p = (x_0, y_\infty)p = (x_0 p, y_\infty)$ is proximal to $x_\infty = (x_0, y_\infty)$ and hence $x_0 p$ is proximal to x_0 which implies that p belongs to some K_i.

(e) Next we show that if $\alpha \in F \setminus A$ then for each i, $K_i \cap \alpha K_i = \emptyset$. To see that we observe that if p and q are both in K_i and $q = \alpha p$ then there are $v_1, v_2 \in J_i$ such that $x_0 p = x_0 v_1$ and $x_0 q = x_0 v_2$. This implies that $x_0 p$ and $x_0 q$ are proximal (e.g. $(x_0 p) v_1 = (x_0 v_1) v_1 = x_0 v_1$ and also $(x_0 q) v_1 = (x_0 v_2) v_1 = x_0 v_1$, since v_1 and v_2 are in the same minimal ideal). On the other hand if $v \in J$ is such that $pv = p$ then also $qv = \alpha p v = \alpha p = q$ and thus $x_0 p$ and $x_0 q$ can be proximal only if they are equal. Therefore $x_0 p = x_0 q = x_0 \alpha p$ and $(x_0 \alpha p) p^{-1} u = x_0 \alpha = x_0$ i.e. $\alpha \in A$. This is a contradiction and we conclude that $K_i \cap \alpha K_i = \emptyset$.

(f) We now show that the cosets space F/A is finite. If not then we can find elements $\alpha_1, \cdots, \alpha_n$ in F such that for every $1 \leqq i \leqq k \leqq n$

$$\alpha_k \cdots \alpha_{i+1} \alpha_i \quad \text{is not in } A.$$

It now follows easily from (e) that

$$L = K \cap \alpha_n (K \cap \cdots (K \cap \alpha_2 (K \cap \alpha_1 K)) \cdots)$$

is empty. On the other hand by (c) L is a residual subset of $\overline{Fw}$, a contradiction. Thus F/A is finite and so is $x_\infty F$.

(g) Since ϕ is RIC and since $x_\infty F$ is finite

$$\phi^{-1}(y_\infty p) = x_\infty F \circ p = x_\infty Fp$$

for every $p \in M$ and it follows that ϕ is a distal homomorphism. Now K is dense in $\overline{Fw}$ and therefore $x_\infty K$ is dense in $\overline{x_\infty Fw}$. Since $\overline{Fw} \subseteq \overline{F}$, $\overline{x_\infty Fw} \subseteq \phi^{-1}(y_\infty)$ and $x_\infty K \subseteq \phi^{-1}(y_\infty)$. By the distality of ϕ, $x_\infty K = \{x_\infty\}$ and it follows that $\overline{x_\infty Fw} = \{x_\infty\}$, $x_\infty Fw = \{x_\infty\}$ and multiplying on the right by u we have $x_\infty F = \{x_\infty\}$, i.e. $F \subseteq A$. Therefore $F = A$ and by (b) the proof is completed.

We do not know whether a metric minimal flow whose enveloping semigroup contains only a countable number of minimal ideals is necessarily PI. However the converse of this statement (and therefore also the converse of Theorem 1) does not hold, as one can see from the next example of a metric, two-steps, strictly PI-flow whose enveloping semigroup contains an uncountable number of minimal ideals.

EXAMPLE. Let $T = SL_2(\mathbf{R})$ be the group of 2×2 real matrices with determinant one. This group acts naturally on the space Y of all lines through the origin of $\mathbf{R}^2$, as well as on the space X of all rays emanating from the origin of $\mathbf{R}^2$. (We think of $\mathbf{R}^2$ as the space of all 1×2 real row vectors and the actions in question are those induced by the product of a row vector with a matrix.) The action of T on the compact spaces X and Y is transitive hence also minimal.

Now the map $(XT) \xrightarrow{\phi} (Y, T)$ which sends a ray onto the line which contains it is a homomorphism of flows and it is easy to see that this is an isometric homomorphism. Also it is easy to check that (Y, T) is a proximal flow (i.e. a proximal extension of the trivial flow). It thus follows that (X, T) is a two steps strictly PI-flow.

Identifying X with the unit circle in $\mathbf{R}^2$ one can show that the minimal ideals of $E(X)$ are in one-to-one correspondence with the partitions of X into two complementary half closed arcs in the following way. If J and J' are two such complementary half closed arcs then the elements of the corresponding minimal ideal I are in one-to-one correspondence with the points of X. If $x \in J$ then the corresponding element $v_x \in I$ maps all of J onto x and all of J' onto its antipodal point (thus in this case v_x is an idempotent). If $x \in J'$ then the corresponding $v_x \in I$ maps all of J onto x and all of J' onto the antipodal point of x.

We conclude that $E(X)$ has $2^{\aleph_0}$ different minimal ideals.

As an easy corollary of Theorem 1 we obtain the following theorem, which extends [6, Prop. 3.1].

THEOREM 2. *Let T be a nilpotent group and let X be a non-trivial minimal weakly mixing flow. Then the enveloping semigroup of X contains infinitely many minimal ideals.*

PROOF. By Theorem 1, X is a PI-flow if $E(X)$ contains only finitely many minimal ideals. Now every minimal flow of a nilpotent group is incontractible (i.e. a RIC-extension of the trivial flow; see [5]) and by [1, 8.10] this implies that if X is PI then X has a non-trivial equicontinuous homomorphic image. Finally, since a homomorphic image of a weakly mixing flow is weakly mixing it is clear that the only equicontinuous homomorphic image of a weakly mixing flow is the trivial one. This completes the proof.

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